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The conditions for the unique solvability of the equations of the dynamics of systems with friction☆

A.P. Ivanov

Moscow, Russia

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ABSTRACT

The problem of determining the generalized accelerations and reactions of constraints in systems with dry friction is investigated. The necessary and sufficient conditions for the unique solvability of the problem are obtained, applicable for cases of sliding and static friction. A geometrical approach is used, based on the introduction of a certain auxiliary parameter space divided into non-overlapping regions in terms of the number of possible types of motion. In each of these regions there are explicit expressions for the accelerations and reactions, which enable us, using piecewise-smooth mapping, to express, from the equations of motion, the generalized forces in terms of the parameters. The solution of the problem is equivalent to inverting the given mapping. A number of examples are given.

The problem considered in this paper dates back to the work of Painlevé,[1,2] which indicated paradoxical situations where a correct solution of the problem was impossible. The numerous investigations since then have generally been devoted to an analysis of specific mechanical systems with friction. A number of sufficient conditions for correctness have also been obtained.[3–7]

1. Formulation of the problem

Consider a mechanical system whose equations of motion in the generalized coordinates $q \in \mathbb{R}^n$ have the form

$$Aw = Q + R, \quad w, Q \in \mathbb{R}^n \tag{1.1}$$

where $w = \ddot{q}$ are the generalized accelerations, the terms Q include the generalized forces and also the velocity-quadratic forces of inertia, R denotes the reactions of unilateral and non-ideal constraints, and A is the symmetric positive-definite matrix expressing the inertia properties. In Eq. (1.1), the values of q, $\dot{q}$ and Q are as specified, and w and R are unknown.

When the system being considered is subject to ideal bilateral constraints only, $R \equiv 0$ in Eq. (1.1), and the generalized accelerations are defined uniquely. Next in complexity is the case of viscous friction, where there is the explicit relationship

$$R = R(q, \dot{q}) \tag{1.2}$$

For the problem under discussion, this case is similar to the previous one.

A feature of dry friction is the dependence of the friction forces on the normal components of the reactions. When Eq. (1.1) are formulated, these components do not disappear, as in the case of ideal constraints, but occur in the equations as unknown quantities. It is also possible that the values of the normal reactions will be uniquely defined at the stage when Eq. (1.1) are formulated. Here, as earlier, a relationship of form (1.2) occurs. The complication compared with the case of viscous friction is due to the discontinuous nature of this relationship. Problems of this kind can be solved by standard methods.[8–10] A general result was obtained:[11] system (1.1), (1.2) for Coulomb friction enables w to be determined uniquely (the static and sliding friction coefficients were assumed to be equal).

In the present paper it is proposed that contact laws be added to Eq. (1.1), expressed by a system of relations of rank n. This means that the unknown quantities w_i and R_i ($i = 1, \ldots, n$) can be expressed in terms of the independent parameters $\pi \in \Pi = \mathbb{R}^n$ by means of the

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E-mail address: apivanov@orc.ru.

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formulae

$$(w, R) = \Phi_j(\pi), \quad \pi \in \Omega_j; \quad j = 1, \dots, s \tag{1.3}$$

where $\Phi_j : \Omega_j \to \mathbb{R}^{2n}$ are differentiable functions. The regions Ω_j have piecewise-smooth bounds and do not overlap, and the combination of their closures is equal to Π. Furthermore, the mapping $\Phi : \Pi \to \mathbb{R}^{2n}$, equal to Φ_j in each of the regions Ω_j, is continuous.

The traditional method for solving system (1.1), (1.3) for specified generalized forces Q consists of the successive solution of s algebraic systems with subsequent checking of inclusions

$$(w, R) \in \Phi_j(\Omega_j); \quad j = 1, \dots, s$$

(see Ref. 11). The reverse approach is used below: from system (1.1), (1.3) we express Q in terms of π:

$$Q = Aw - R = \psi(\pi) \tag{1.4}$$

The problem of the solvability of system (1.1), (1.3) reduces to considering the properties of a continuous piecewise-differentiable mapping ψ. The surjectivity of this mapping is equivalent to the existence of a solution, and its injectivity is equivalent to its uniqueness.

Example 1. For a system with one degree of freedom and unilateral contact

$$q_1 \ge 0 \tag{1.5}$$

in the case $q_1 = 0$ and $\dot{q}_1 = 0$, the following conditions of complementarity[12] are satisfied

$$w_1 \ge 0, \quad R_1 \ge 0, \quad w_1 R_1 = 0 \tag{1.6}$$

and we have $\Pi = \mathbb{R}$ and $s = 2$. Formulae (1.3), corresponding to conditions (1.6), are (the given representation is non-unique):

$$\Omega_1 = \{\pi | \pi < 0\}, \quad \Phi_1(\pi) = (0, -\pi); \quad \Omega_2 = \{\pi | \pi > 0\}, \quad \Phi_2(\pi) = (\pi, 0) \tag{1.7}$$

Example 2. Sliding friction is described by the formula

$$T = -\mu |N| \upsilon / |\upsilon| \tag{1.8}$$

where N and T are the normal and tangential components of the reaction, μ is the coefficient of friction and v is the sliding velocity, which is a known function of the generalized coordinates and velocities. If the contact is unilateral, then constraint (1.5) exists and conditions similar to (1.6) (with R_1 replaced by N) are satisfied. The dependence of w_1 and N on the parameter π is described by formulae similar to (1.7). In the case of bilateral contact, $w_1 \equiv 0$; consequently

$$\Omega_1 = \{\pi | \pi < 0\}, \quad \Omega_2 = \{\pi | \pi > 0\}; \quad \Phi_1(\pi) = \Phi_2(\pi) = (0, \pi)$$

The retention in this case of two regions of definition is related to the general idea of plotting differentiable mappings and with the presence of a sign of absolute magnitude in formula (1.8).

Example 3. For static friction, $v = 0$ and formula (1.8) cannot be used. If $w \ne 0$, i.e., sliding begins at the instant considered, then, assuming in formula (1.8) that

$$\upsilon = w\Delta t + o(\Delta t)$$

we will arrive at the relations[1]

$$T = -\mu^* |N| w / |w|, \quad |T| \le \mu^* |N| \ \text{при} \ w = 0 \tag{1.9}$$

where μ^* is the static coefficient of friction (when the adhesion component of the friction is taken into account forces, it must be assumed that $\mu < \mu^*$). Suppose that, according to the conditions of contact, $w \in \mathbb{R}$, i.e. sliding is only possible along a certain straight line, with a unilateral frictional contact. We will assume that the coordinate $q_2 \ge 0$ corresponds to the normal direction and that $q_1 \in \mathbb{R}$ corresponds to the tangential direction. The laws (1.5) and (1.9) include four cases: the start of sliding to the left or to the right, rest, and detachment. Formulae (1.3) corresponding to them are as follows (see Fig. 1):

$$\begin{aligned}
&\Omega_1 = \{\pi | \pi_1 < \pi_2 < 0\}, \quad \Phi_1(\pi) = (\pi_1 - \pi_2, 0, -\mu^* \pi_2, -\pi_2) \\
&\Omega_2 = \{\pi | -\pi_1 < \pi_2 < 0\}, \quad \Phi_2(\pi) = (\pi_1 + \pi_2, 0, \mu^* \pi_2, -\pi_2) \\
&\Omega_3 = \{\pi | \pi_1 < 0, |\pi_1| < -\pi_2\}, \quad \Phi_3(\pi) = (0, 0, -\mu^* \pi_1, -\pi_2) \\
&\Omega_4 = \{\pi | \pi_1 > 0, \pi_1 \in \mathbb{R}\}, \quad \Phi_4(\pi) = (\pi_1, \pi_2, 0, 0)
\end{aligned} \tag{1.10}$$

2. General conditions of existence and uniqueness

In a general formulation, we will consider the continuous mapping $\psi : \Pi \to \mathbb{R}^n$, which is continuously differentiable in each of the closed regions $\bar{\Omega}_j$ possessing the above properties. In each of the regions Ω_j we will define the Jacobian matrix $J_j(\pi)$, the elements of which are partial derivatives

$$\partial \psi_r / \partial \pi_l \ (r, l = 1, \dots, n)$$

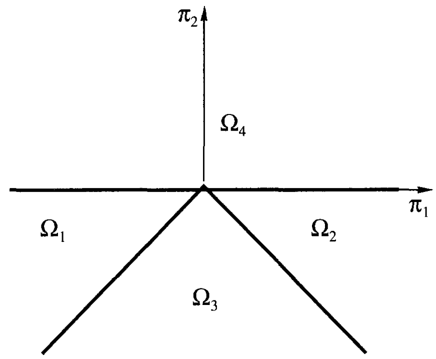

Fig. 1.

Theorem. *Suppose the following conditions are satisfied:*

1) $\det J_j(x) > 0, \quad x \in \Omega_j; \quad j = 1, \ldots, s$ (2.1)
2) *The elements of the matrices* $J_j^{-1}(x)$ *are uniformly bounded in P.*
3) *At the boundary points of the regions* Ω_j, *the mapping* ψ *is one-to-one.*

Then ψ *is a bijection. If the Jacobians* (2.1) *can take values of both signs, then the mapping* ψ *is not one-to-one.*

Proof. If $s = 1$, we obtain the well-known result concerning global diffeomorphism.[13] If $s > 1$, then, according to the principle of region conservation, $\psi(\Omega_j)$ are also regions, where internal points are mapped into internal points and the boundary points are mapped into boundary points.

We will show that, in each of the regions Ω_j, the mapping ψ is injective. Since

$$\psi(x + \Delta x) - \psi(x) = J_j(x)\Delta x + o(\Delta x)$$

then taking account of condition 2, we obtain

$$\|\Delta x\| \le C\|\psi(x + \Delta x) - \psi(x)\| + o(\Delta x) \tag{2.2}$$

where C is a certain constant. If $\psi(x_1) = \psi(x_2) = y^*$ for certain $x_1,\ x_2 \in \Omega_j$, then, when y^* changes, two branches of the inverse images are retained, and here, by virtue of inequality (2.2), they are unable to merge. Continuing y^* to the boundary of the region Ω_j, we arrive at a conclusion concerning the non-uniqueness of the mapping ψ at the boundary, contradicting condition 3.

The next step is to prove the equation

$$\psi(\partial\Omega_j) = \partial\psi(\Omega_j) \tag{2.3}$$

according to which the image of the boundary is equal to the boundary of the image (from the principle of region conservation it follows only that the left-hand side of Eq. (2.3) is a subset of the right-hand side). If $\bar{y} \in \partial\psi(\Omega_j)$, then $\bar{y} = \lim y_i,\ y_i = \psi(x_i),\ x_i \in \Omega_j$. Since the sequence $\{y_i\}$ is fundamental, it follows that, owing to inequality (2.2), the sequence of inverse images $\{x_i\}$ is also fundamental, and here $\lim x_i = \bar{x} \in \bar{\Omega}_j$. However, if $\bar{x} \in \Omega_j$, then it is also necessary that $\bar{y} \in \psi(\Omega_j)$. Consequently, $\bar{x} \in \partial\Omega_j$.

The meaning of inequalities (2.1) still requires clarification. Suppose x_0 is a regular point of the common boundary of regions Ω_1 and Ω_2. According to the conditions, for any tangential direction t we have

$$\partial\psi(\Omega_1)/\partial t = \partial\psi(\Omega_2)/\partial t$$

As regards the normal direction n, on account of conditions (2.1) the vectors $\partial\psi(\Omega_1)/\partial n$ and $\partial\psi(\Omega_2)/\partial n$ lie on the same side of the tangential plane. Since the vector n is directed from point x_0 towards only one of the regions Ω_1 and Ω_2, the images $\psi(\Omega_1)$ and $\psi(\Omega_2)$ do not overlap. On the other hand, a change in sign of the Jacobian at the boundary would indicate their overlapping. Finally, a change in sign of the Jacobian at an internal point of region Ω_j denotes the presence of a singularity and the absence of uniqueness. The theorem is proved.

Corollary 1. *In the case where all mappings* $\psi(\Omega_j)$ *are linear, conditions* (2.1) *are necessary and sufficient for a unique solution of the equations of motion to exist for any* $Q \in \mathbb{R}^n$.

In fact, in this case the matrices J_j are constant, and the conditions of the theorem follow from conditions (2.1). Note that uniqueness also occurs in the case where all Jacobians in conditions (2.1) are negative. The use of coordinate systems with a positive orientation is more customary.

Example. Suppose that in the system there is unilateral contact (1.5) with friction (1.8), with a tangential velocity component $\dot{q}_2 \neq 0$. After substituting relations (1.7) and (1.8) into system (1.1), we obtain a system with a new matrix $\bar{A}$. The criterion for unique solvability of the system, equivalent to conditions (2.1), is that the matrix $\bar{A}$ belongs to the class of P-matrices,[14] which means that all the principal minors are positive. Similarly, it is possible to investigate systems with several unilateral or bilateral contacts and sliding friction.[15]

3. Systems with fixed lines of action of the friction forces

Suppose that, at each point of frictional contact, the tangential and normal directions are uniquely defined, which enables the Amonton–Coulomb law to be used in form (1.8) or (1.9). The contacts are not necessarily kinematically independent, i.e., sliding at one of the points may preclude rest at another point, and so on. In any case, each possible movement of the system leads either to loss of contact or to the start of sliding in one of the two possible directions, or to retention of zero slippage (rolling).

We will show that, for systems of the type considered, linear coordinate mappings Φ_j can be chosen. We will divide the space $\Pi = \mathbb{R}^n$ into regions Ω_j according to the possible types of motion, and we will introduce into Π a Cartesian system of coordinates. Each of the regions Ω_j is a polyhedron, every face of which contains an origin of coordinates and corresponds to a change in state of one of the contacts. If all the contacts are independent of each other, the mappings Φ_j are expressed by linear formulae of type (1.7) or (1.10). The presence of kinematic relations leads to functional relations between motions or velocities, and also between reactions. If the mechanical system considered has finite or linear differential constraints, then these functional relations are also linear. The following result was obtained.

Corollary 2. *Suppose all frictional contacts belong to one of two groups: either the sliding velocity is non-zero or it is zero but sliding can begin only along some straight line (which differs for the different contacts). Conditions* (2.1) *are then necessary and sufficient for a unique solution of the equations of motion to exist for any* $Q \in \mathbb{R}^n$.

Remark 1. For the given problem, sufficient conditions of existence and uniqueness were obtained earlier[3] on the basis of additional variables and a corresponding iterational algorithm. These conditions require the Jacobian matrices, similar to those considered here, to belong to the class of *P*-matrices. In accordance with the above results, the given requirements are slightly exaggerated.

Example 1. The problem of the plane-parallel motion of a rod with one of its ends *C* in contact with a horizontal rough support has a history stretching back more more than 100 years.[2,3,16] The coordinates of point *C* will be taken as the generalized coordinates q_1, q_2, and the angle of inclination of the rod to the support will be taken as q_3. Considering the mass and radius of inertia of the rod to be unique, and putting $l = |CG|$ (where *G* is the center of mass of the rod), we will formulate system (1.1) using the principal theorems of dynamics:

$$\begin{aligned} &w_1 - hw_3 = Q_1 + R_1, \quad w_2 + aw_3 = Q_2 + R_2 \\ &w_3 = Q_3 - aR_2 + hR_1 + R_3, \quad h = l\sin q_3, \quad a = l\cos q_3 \end{aligned} \tag{3.1}$$

where $w_i = \ddot{q}_i \, (i = 1, 2, 3)$.

The contact conditions contain a unilateral constraint, and also the law of dry friction:

$$q_2 \ge 0, \quad T = -\mu^* N \operatorname{sign} v_1, \quad |T| \le \mu^* N \tag{3.2}$$

(this last inequality corresponds to no slippage). We will assume that, at a given instant of time, $q_2 = 0$ and $v_2 = 0$, i.e. the vertical velocity component of the contact point is zero. We will take $\Pi = \mathbb{R}^3$ as the parameter space, and the mapping ψ, which figures in the theorem, will be specified by formulae (1.10), where in all cases it must be assumed that

$$w_3 = \pi_3, \quad R_1 = T, \quad R_2 = N, \quad R_3 = 0$$

Substituting these formulae into system (3.1), we obtain for the mapping ψ

$$Q_1 = w_1 - hw_3 - R_1, \quad Q_2 = w_2 + aw_3 - R_2, \quad Q_3 = w_3 + aR_2 - hR_1 \tag{3.3}$$

Taking expressions (1.10) into account, we will write the Jacobian matrices in the form

$$J_{1,2} = \left\| \begin{matrix} 1 & \pm(\mu - 1) & -h \\ 0 & 1 & a \\ 0 & \pm h\mu - a & 1 \end{matrix} \right\|, \quad J_3 = \left\| \begin{matrix} \mu & 0 & -h \\ 0 & 1 & a \\ h\mu & -a & 1 \end{matrix} \right\|, \quad J_4 = \left\| \begin{matrix} 1 & 0 & -h \\ 0 & 1 & a \\ 0 & 0 & 1 \end{matrix} \right\| \tag{3.4}$$

where the superscript on the variable μ has been omitted.

Conditions (2.1) reduce to the unique inequality[2,16]

$$1 + a^2 > |a| h\mu \tag{3.5}$$

We will give a geometrical interpretation of the solution of system (3.1) when condition (3.5) is satisified; we will assume that $a > 0$. Omitting the variable w_3 from Eq. (3.3), we will present them in the form

$$\begin{aligned} &\bar{Q}_1 = Q_1 + hQ_3 = w_1 - (1 + h^2)R_1 + ahR_2 \\ &\bar{Q}_2 = Q_2 - aQ_3 = w_2 + ahR_1 - (1 + a^2)R_2 \end{aligned} \tag{3.6}$$

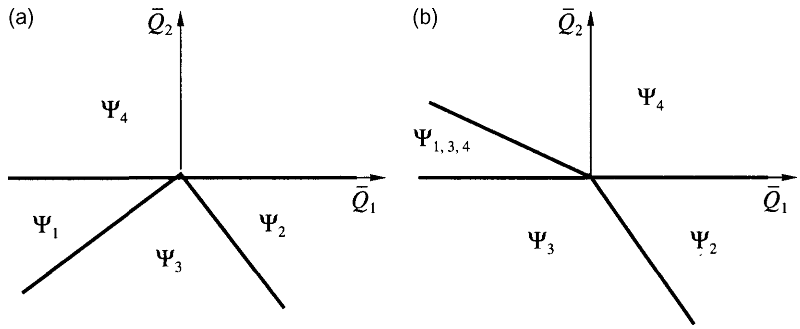

Fig. 2.

In the $(\bar{Q}_1, \bar{Q}_2)$ plane, the images of the regions (1.10) are bounded by the lines

$$\bar{Q}_2 = 0, \quad \bar{Q}_2 = \bar{Q}_1 \frac{(1+h^2)\mu - ah}{1+a^2-ah\mu} \; (\bar{Q}_2 < 0)$$
$$\bar{Q}_2 = -\bar{Q}_1 \frac{(1+h^2)\mu + ah}{1+a^2+ah\mu} \; (\bar{Q}_2 < 0) \tag{3.7}$$

(Fig. 2a). Depending on the position of the point $(\bar{Q}_1, \bar{Q}_2)$ in a particular region $\psi(\Omega_j)$ $(j = 1, \ldots, 4)$, it is possible to determine the form of motion in system (3.1) and, accordingly, eliminate surplus unknowns. Hence, an exhaustive search of all possibilities is no longer necessary.

Suppose now that $a > 0$ and that the condition (3.4) is not satisfied. The second of the lines (3.7), which is the boundary between ψ_1 and ψ_3, then lies in the upper half-plane (Fig. 2b). Consequently, region ψ_1 is the overlap of regions ψ_3 and ψ_4. When the point $(\bar{Q}_1, \bar{Q}_2)$ falls in region ψ_1, three solutions of system (3.1) are possible, corresponding to detachment, rest, and sliding – the so-called non-uniqueness paradox.[1–3]

Finally, for the values of the parameters for which relation (1.5) changes into an equality, the first two lines (3.7) merge, and the region ψ_1 disappears. For values $(\bar{Q}_1, \bar{Q}_2) \in \Psi_j$ $(j = 2, 3, 4)$, the motion is defined uniquely. However, if

$$\bar{Q}_1 < 0, \quad \bar{Q}_2 = 0$$

then system (3.6) has a continuum of solutions corresponding to sliding of the rod to the left at an arbitrary acceleration $w_1 \in (\bar{Q}_1, 0)$:

$$w_1 = \bar{Q}_1 + ((1+h^2)\mu - ah)R_2, \quad w_2 = 0, \quad R_1 = \mu R_2 \geq 0$$

Example 2. We will modify the previous example, considering the constraint between the rod and the support to be bilateral. Here, in Eq. (3.1) it must be assumed that $w_2 \equiv 0$, and the normal reaction N may be both positive and negative. The number of components into which the space Π is divided increases to 6: to the regions Ω_1, Ω_2 and Ω_3 are added the three parts into which the region of detachment Ω_4 breaks down. These parts – $\Omega'_3 = \{\pi | \pi_2 > 0, \pi_2 > |\pi_1|\}$ (rolling), $\Omega'_1 = \{\pi | -\pi_1 > \pi_2 > 0\}$ (sliding to the left) and $\Omega'_2 = \{\pi | \pi_1 > \pi_2 > 0\}$ (sliding to the right) – are centrally symmetric with the first three regions. For them, the mappings Ω'_1, Ω'_2 and Ω'_3 are specified by formulae similar to (1.10) but with a change in the sign of μ. Consequently, the conditions of existence and uniqueness (3.5) also remain unchanged.

Example 3. Adding to the system from Example 1 a restriction in the form of a vertical wall (Fig. 3) against which the end C of the rod rests, we will obtain the well-known problem of the equilibrium of a ladder upon which a person is standing (see, for example, Refs 17 and 18). Suppose, to begin with, that the wall is smooth, and the constraint between the wall and the rod is bilateral. Then, q_1 and q_2 can be

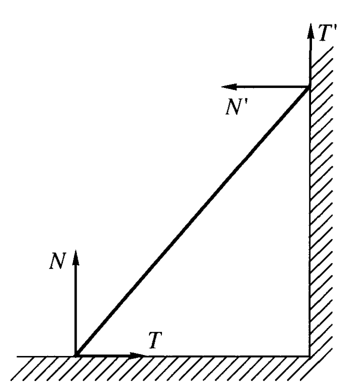

Fig. 3.

taken as generalized coordinates, and the angle q_3 is defined by the formula

$$q_1 + (l + l')\cos q_3 = d, \quad l' = |GC'| \tag{3.8}$$

where d is the abscissa of the surface of the wall. In Eq. (3.1), the reaction of the wall N' will be added:

$$w_1 - hw_3 = Q_1 + R_1 - N', \quad w_2 + aw_3 = Q_2 + R_2$$
$$w_3 = Q_3 - aR_2 + hR_1 + h'N', \quad h' = l'\sin q_3 \tag{3.9}$$

The quantity w_3 can be expressed in terms of w_1, twice differentiating formula (3.8), and then N' can be omitted from the third equation of (3.9). As a result we obtain

$$\tilde{Q}_1 = \xi^{-1}w_1 - (h + h')R_1 + aR_2, \quad \tilde{Q}_1 = hQ_1 + Q_3$$
$$\tilde{Q}_2 = w_2 + a\xi R_1 - (1 + a\eta)R_2, \quad \tilde{Q}_2 = Q_2 - \eta\tilde{Q}_1$$
$$\xi = \frac{h + h'}{h'^2 + 1}, \quad \eta = \frac{a}{h'^2 + 1} \tag{3.10}$$

Substituting relations (1.10) into formulae (3.10), we obtain the following expressions for the Jacob matrices

$$J_{1,2} = \left\| \begin{matrix} \xi^{-1} & \mp\xi^{-1} \pm (h + h')\mu - a \\ 0 & \mp a\xi\mu + \dfrac{1 + a^2 + h'^2}{h'^2 + 1} \end{matrix} \right\|, \quad J_3 = \left\| \begin{matrix} (h + h')\mu & -a \\ -a\xi\mu & 1 + a\eta \end{matrix} \right\|, \quad J_4 = \left\| \begin{matrix} \xi^{-1} & 0 \\ 0 & 1 \end{matrix} \right\|$$

Consequently, conditions (2.1) reduce to the inequality

$$a(h + h')\mu < 1 + a^2 + h'^2 \tag{3.11}$$

If this inequality has a positive meaning, then $\det J_1 < 0$. The arrangement of the regions Ψ_j ($j = 1, \ldots, 4$) is then similar to that presented in Fig. 2b. Here, for certain values of the external forces, system (3.10) has three solutions corresponding to immobility of the rod, its sliding to the left and its detachment from the support.

We will investigate the following problem: will the "ladder" remain at rest under specified external forces? For a positive solution it is necessary to establish that the point $(\tilde{Q}_1, \tilde{Q}_2)$ lies in the region Ψ_3 but does not lie in any of the regions $\Psi_j, j \neq 3$. Suppose gravity P alone is acting on the rod. Then

$$Q_1 = 0, \quad Q_2 = -P, \quad Q_3 = 0 \tag{3.12}$$

whence

$$\tilde{Q}_1 = 0, \quad \tilde{Q}_2 = -P$$

If inequality (3.11) is violated, then, as can be seen from Fig. 2b, the conditions of rest given above are satisfied. If this inequality is satisfied, however, it is necessary to ascertain the quarter in which the boundary between Ψ_1 and Ψ_3 lies (see Fig. 2a). Assuming in equalities (3.10) that

$$w_1 = w_2 = 0, \quad R_1 = \mu R_2$$

we obtain the following equation for this boundary

$$\tilde{Q}_1 = (a - (h + h')\mu)R_2, \quad \tilde{Q}_2 = \frac{a(h + h')\mu - 1 - a^2 - h'^2}{h^2 + 1}R_2$$

Here, taking inequality (3.11) into account, we have $\tilde{Q}_2 < 0$, and therefore the rod begins to slide when $\tilde{Q}_1 > 0$, while it remains at rest when $\tilde{Q}_1 \le 0$.

The following conclusion can be drawn: the condition

$$(h + h')\mu \ge a \tag{3.13}$$

is necessary and sufficient for the rod to remain at rest. Note that, as a person climbs up the ladder, the system parameters change in such a way that the left-hand side of inequality (3.13) remains unchanged while the right-hand side increases. Here, the coefficient of friction necessary to prevent slippage of the ladder also increases.

Example 4. We will assume the wall to be rough, and the constraint between it and the rod to be unilateral:

$$q_1 + a' \le d, \quad a' = l'\cos q_3$$

where q_1 and q_2 are now the coordinates of the center rod. The reactions of the wall N' and T' will be added to the acting forces, so that the equations of motion will take the form

$$w_1 = Q_1 + T - N', \quad w_2 = Q_2 + N + T'$$
$$w_3 = Q_3 + a'T' - aN + hT + h'N' \tag{3.14}$$

Likewise, it is not difficult to set up a group of relations similar to (3.2) and (1.6) for the contact of the rod with the wall:

$$q_1 + a' \le d, \quad N' \ge 0, \quad (q_1 + a' - d)N' = 0$$
$$T' = -\mu' N' \operatorname{sign}(\dot{q}_2 + a'\dot{q}_3), \quad |T'| \le \mu' N'$$

where μ' is the coefficient of friction at this contact.

In the coordinate space $\Pi = \mathbb{R}^3$ there are ten regions Ω_j bounded by planes $\pi_2 = 0$, $\pi_3 = 0$ (corresponding to detachment of one of the ends of the rod from the support), $\pi_3 = -|\pi_1|$ in the region $\pi_2 \ge 0$ and $\pi_2 = -|\pi_1|$ in the region $\pi_3 \ge 0$ (corresponding to detachment at one of the points) and also $\pi_2 + \pi_3 0 = -|\pi_1|$ in the region $\pi_2 \le 0$, $\pi_3 \le 0$ (corresponding to double contact). The mappings Ω_j are plotted by analogy with (1.10), and here, in the regions of double contact Ω_8 (sliding to the left) and Ω_9 (sliding to the right), we have

$$\Phi_8 = (h'\vartheta, a\vartheta, \vartheta, -\pi\mu_2 + \pi_3, -\pi_2 - \mu'\pi_3, (h\mu - a)\pi_2 + (h' + a'\mu')\pi_3)$$
$$\vartheta = \pi_1 - (\pi_2 + \pi_3) \tag{3.15}$$

while the formula for Ω_9 is obtained from (3.15) by replacing the values of the coefficients of friction with the opposite values, with a simultaneous change in sign from minus to plus in the expression for ϑ.

Among the possible motions of the rod there are those for which it slides over the support, detaching itself from the wall, and therefore condition (3.5) remains in force. Furthermore, it can slide along the wall, detaching itself from the support; for this case, by analogy with condition (3.5), we obtain the condition

$$a'|h'|\mu' < 1 + h'^2 \tag{3.16}$$

There remain motions where both ends of the rod slide over the supports. Substituting expression (3.15) into Eq. (3.14) we obtain

$$J_8 = \left\| \begin{matrix} h & \mu & -1 \\ a & 1 & \mu' \\ 1 & h\mu - a & a'\mu' + h' \end{matrix} \right\|$$

$$\det J_8 = 1 + a^2 + h'^2 + (1 - hh' - aa')\mu\mu' - a(h + h')\mu + h'(a + a')\mu'$$

The matrix J_9 can be found from J_8 by replacing the values of μ and μ' by the opposite values. The conditions for unique solvability are described by the system of inequalities (3.5), (3.16), and also by the inequality

$$1 + a^2 + h'^2 + (1 - hh' - aa')\mu\mu' > |a(h + h')\mu - h'(a + a')\mu'| \tag{3.17}$$

For the case of a smooth wall ($\mu' = 0$), inequality (3.16) is satisfied automatically, and condition (3.17) is identical with (3.11).

We will check to see whether the system is in equilibrium under gravity. For this, it is necessary to establish that the point with coordinates (3.12) lies in the region Ψ_{10} but does not lie in regions Ψ_8 (sliding to the left) and Ψ_4 (detachment from the support). Note that the condition of equilibrium of the ladder, obtained earlier by a geometrical method (see reference 17, §193) and expressed by the inequality

$$(h + h' + a'\mu')\mu > a \tag{3.18}$$

means that $Q \in \Psi_{10}$. Our calculations indicate that, if inequality (3.16) has the opposite meaning, with $h' < 0$, then inequality (3.18) is insufficient for the system to be in equilibrium: here, condition (3.17) is also violated, and $Q \in \Psi_4 \cap \Psi_8$, i.e., along with equilibrium, slippage of the ladder is possible. If $h' < 0$, condition (3.18) is necessary and sufficient for equilibrium.

From a practical viewpoint, the inequality $h' < 0$ means that the centre of mass of the person lies above the upper end of the ladder, the latter being much lighter than the person.

4. Systems with an uncertain direction of sliding

In the general case, in formulae (1.8) and (1.9) the relative velocities v, the accelerations w and the component reactions T are vectors lying in tangential planes determined for each of the contact points. For contacts where $v \neq 0$, the directions of the reactions at a given instant are known, and the above methods can be used. Greater difficulties arise if there are contact points at which the relative velocity is zero and the directions in which sliding can begin occupy a tangential plane.

In coordinate space Π, two regions separated by a conical surface correspond to two versions of law (1.9) (the start of sliding and rolling). In the case of unilateral contact, a third version is also possible: weakening of the constraint. Division of the space P^3 into three regions can be assigned to such contact: Ω_1: $\{\pi_3 > 0\}$ – detachment; $\Omega_2 : \{\pi_3 < 0, \pi_1^2 + \pi_2^2 < \pi_3^2\}$ – rolling; $\Omega_3 : \{\pi_3 < 0, \pi_1^2 + \pi_2^2 > \pi_3^2\}$ – sliding. For region Ω_2, the mapping (1.3) is linear: it can be assumed that $w = 0$ and $(T, N) = -(\mu\pi_1, \mu\pi_2, \pi 3)$. In the case of detachment, $w = \pi$

and $(T, N) = 0$. In region Ω_3, formula (1.9) is non-linear in accelerations, and it is more convenient to describe it in a cylindrical system of coordinates, assuming that

$$N = -\pi_3, \quad T = \mu\pi_3(\cos\varphi, \sin\varphi)$$
$$w_t = \rho(\cos\varphi, \sin\varphi), \quad w_n = 0, \quad \rho = (\pi_1^2 + \pi_2^2 - \pi_3^2)^{1/2} \tag{4.1}$$

The subscripts t and n correspond to the tangential and normal components of the vector.

In accordance with definition (4.1), various points of the surface of the cone, depending on the direction of slippage, correspond to the particular point $w_t = 0$ of law (1.9), which enables global continuity of the mapping ψ to be achieved.

Unlike the previous section, formulae (4.1) lead to a mapping ψ that is linear in π_3 and ρ but non-linear in φ. Because of this, checking of the conditions of the theorem is considerably more complicated. Suppose, to begin with, that the frictional contacts in the system are independent of each other. Then, each of the components of the mapping ψ is linear in N_i and ρ_i, with the coefficients periodic with respect to φ_i (the subscript i corresponds to various points of contact). When compiling the Jacob matrix $J_j(\pi)$, three rows will correspond to each of the contact points, corresponding to derivatives with respect to the variables N_i, ρ_i and φ_i. The elements of two of these rows depend only on φ_i, while the elements of the third row are linear functions of N_i and ρ_i. Consequently, $\det J_j(\pi)$ is a polynomial in N and ρ, with the periodic coefficients linear in each of the variables N_i and ρ_i. In view of the independence of these variables, inequalities (2.1) mean that all the coefficients of the given polynomial are positive for all φ_i. On the other hand, condition 2 of the theorem also follows from this, as the cofactors of the matrix $J_j(\pi)$ are also polynomials of an order not exceeding the order of the polynomial $\det J_j(\pi)$. Note that degeneracy of the Jacob matrices at the boundary points $N_i = R_i = 0$ is possible. The uniqueness at these points follows from the theorem of the mechanical energy and dissipative nature of friction. Finally, the third condition of the theorem follows from the injectiveness of the mapping set by formulae (4.1). The following assertion is proved.

Corollary 3. *Suppose all three-dimensional frictional contacts in system* (1.1) *are independent. Then, satisfaction of inequalities* (2.1) *in the general case is necessary and sufficient for a unique solution of the equations of motion to exist for any* $Q \in \mathbb{R}^n$.

Remark. The exceptions are cases where just one of the coefficients of the polynomial $\det J_j(\pi)$ may take zero values but does not take negative values. In this case the second condition of the theorem does not follow from the first condition and requires separate checking.

Example. Consider a rigid body in contact with a stationary rough support at a single point C. At a given instant of time, the velocities of all points of the body are zero. Using the basic theorems of dynamics, we will write Eq. (1.1) in the form

$$m\dot{\upsilon}_G = F + R, \quad I\dot{\omega} = M + r_C \times R \tag{4.2}$$

where m is the mass of the body, I is the inertia tensor of the body, υ_G and ω are the velocity of the centre of mass G and the angular velocity, $r_C = GC$ is the radius vector of the contact point, and F and M are the principal vector and principal moment of the external forces. With these assumptions, we have the following expression for the velocity of the contact point υ_C

$$\dot{\upsilon}_C = \dot{\upsilon}_G + \dot{\omega} \times r_C \tag{4.3}$$

Substituting expressions (4.2) into formula (4.3), we obtain the equation

$$m\dot{\upsilon}_C = F^* + BR; \quad F^* = F + m(I^{-1}M) \times r_C, \quad BR = R + mI^{-1}(r_C \times R) \times r_C \tag{4.4}$$

It can be shown by direct checking that the matrix B is symmetrical and positive, and Eq. (4.4) can therefore be written in the form (1.1), where

$$A = mB^{-1}, \quad Q = B^{-1}F^*, \quad w = \dot{\upsilon}_C$$

Solving this equation for w and R, it is possible to determine $\dot{\omega}$ uniquely from the second equation of (4.2). Consequently, the problem under discussion reduces to considering Eq. (1.1) in $\mathbb{R}^3$ under condition of complementarity (1.6) in the normal direction and with friction law (1.9).

In regions Ω_1 and Ω_2 the mapping ψ is linear, and inequalities (2.1) are satisfied. In region Ω_3 this mapping, taking relations (4.1) into account, is expressed by the formula

$$Q = \rho A \begin{Vmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{Vmatrix} + \pi_3 \begin{Vmatrix} -\mu\cos\varphi \\ -\mu\sin\varphi \\ 1 \end{Vmatrix} \tag{4.5}$$

We will write the Jacob matrix $J_3 = \partial Q/\partial(\rho, \varphi, \pi_3)$ in the form of a set of columns

$$J_3 = (Ae_\varphi, \rho Ae_\beta - \mu\pi_3 e_\beta, e_n - \mu e_\varphi)$$
$$e_\varphi = (\cos\varphi, \sin\varphi, 0)^T, \quad e_\beta = (-\sin\varphi, \cos\varphi, 0)^T, \quad e_n = (0, 0, 1)^T \tag{4.6}$$

We have

$$\det J_3 = D_\rho\rho - \mu D_\pi \pi_3$$
$$D_\rho = (Ae_\varphi \times Ae_\beta, e_n - \mu e_\varphi) = A_{33} - \mu(A_{13}\cos\varphi + A_{23}\sin\varphi), \quad D_\pi = (Ae_\varphi, e_\varphi + \mu e_n) \tag{4.7}$$

where A_{ij} are the cofactors of the matrix A. As the quantity π_3 is negative in the region Ω_3, the conditions of Corollary 3 mean that

$$D_\rho > 0, \quad D_\pi > 0, \quad \forall\varphi \in [0, 2\pi] \tag{4.8}$$

From the geometrical viewpoint, the first inequality of (4.8) means that, in the space Q, the region of detachment does not overlap with the cone of friction, while the second inequality means that the region of sliding does not overlap with this cone. Algebraic checking of the first inequality is simple: owing to the definition of the matrix A, it is equivalent to the relation

$$\mu\sqrt{b_{13}^2 + b_{23}^2} < b_{33}$$

which is similar to inequality (3.5).

We will write the second inequality of (4.8) in the form

$$a_{11}\cos^2\varphi + a_{12}\sin 2\varphi + a_{22}\sin^2\varphi + \mu(a_{31}\cos\varphi + a_{32}\sin\varphi) > 0 \tag{4.9}$$

Using the universal trigonometric substitution

$$t = \operatorname{tg}\frac{\varphi}{2}, \quad \sin\varphi = \frac{2t}{1+t^2}, \quad \cos\varphi = \frac{1-t^2}{1+t^2}$$

inequality (4.9) can be reduced to the algebraic form

$$\begin{gathered} P(t) = p_4t^4 + p_3t^3 + p_2t^2 + p_1t + p_0 > 0 \\ p_4 = a_{11} - \mu a_{31}, \quad p_3 = 2\mu a_{32} - 4a_{12}, \quad p_2 = 4a_{22} - 2a_{11} \\ p_1 = 4a_{12} + 2\mu a_{32}, \quad p_0 = a_{11} + \mu a_{31} \end{gathered} \tag{4.10}$$

The following cubic resolvent corresponds to polynomial (4.10)

$$\begin{gathered} Q(t) = t^3 - 2q_2t^2 + (q_2^2 - 4q_0)t + q_1^2 \\ q_2 = \bar{p}_2 - \frac{3}{8}\bar{p}_3^2, \quad q_0 = \bar{p}_0 - \frac{1}{4}\bar{p}_1\bar{p}_3 + \frac{1}{16}\bar{p}_2p_3^2 - \frac{3}{256}\bar{p}_3^4 \\ q_1 = \bar{p}_1 - \frac{1}{2}\bar{p}_2\bar{p}_3 + \frac{1}{8}\bar{p}_2^2, \quad \bar{p}_i = \frac{p_i}{p_4}, \quad i = 0, 1, 2, 3 \end{gathered} \tag{4.11}$$

The satisfaction of inequality (4.10) for all $t \in \mathbb{R}$ is equivalent to satisfying the inequality $p_4 > 0$ together with the requirement of the presence in the resolvent (4.11) of one negative and two positive roots.[19] The criterion for the presence in polynomial (4.11) of three real roots is the condition for its discriminant to be non-positive:

$$D = -\frac{1}{27}\left(\frac{1}{3}q_2^2 + 4q_0\right)^3 + \frac{1}{4}\left(\frac{2}{27}q_2^3 - \frac{8}{3}q_2q_0 + q_1^2\right)^2 \le 0 \tag{4.12}$$

As the free term of this polynomial is positive, it can have either one or three negative roots. For the second of these cases, to be excluded from consideration, the Hurwitz conditions[20] are satisfied:

$$q_2^2 - 4q_0 > 0, \quad -2(q_2^2 - 4q_0)q_2 > q_1^2 \tag{4.13}$$

Thus, the second of conditions (4.8) consists of inequalities $p_4 > 0$ and (4.12), and also a set of inequalities opposite to (4.13) (bearing in mind that just one of these opposite inequalities is satisfied).

Checking of these conditions in practice is elementary but fairly cumbersome. The following simple inequality, sufficient for validating conditions (4.8) for any φ, may therefore turn out to be useful

$$2\mu\sqrt{a_{31}^2 + a_{32}^2} < a_{11} + a_{22} - \sqrt{4a_{12}^2 + (a_{11} - a_{22})^2} \tag{4.14}$$

Note that determinant (4.7) vanishes for $\rho = \pi_3 = 0$, corresponding to the boundary of region Ω_3. We will show that this equation has no non-zero solutions. Multiplying both sides of this equation by the vector e_φ, we obtain

$$0 = \rho(Ae_\varphi, e_\varphi) - \pi\mu_3$$

which contradicts the inequalities $\rho > 0$ and $\pi_3 < 0$.

We will now consider the case of dependent frictional contacts. Note that here it is not always possible to determine the reactions at each of the contacts. It is a matter solely of searching for the vector R in system (1.1), the components of which are made up of individual reactions. For each three-dimensional contact we will use coordinates of the form (4.1), considering the differentials of these variables to be independent by virtue of the kinematic constraints imposed. The Check to see whether the conditions of the theorem are satisfied is carried out in the same way as in the case of independent contacts.

Example 1. The two point masses m_1 and m_2 are connected by a weightless rod of length $2l$ and pressed against a rough plane with a coefficient of friction μ by forces N_1 and N_2 normal to the plane.[11] The external forces act in the same plane. At the initial instant of time, the system is at rest. The theorems of momentum and moments (relative to the centre of the rod) are expressed by the formulae

$$(m_1+m_2)w_3 = Q_1+R_1, \quad m_1w_1+m_2w_2 = Q_2+R_2, \quad m_2w_2-m_1w_1 = Q_3+R_3$$
$$R_1 = T_{1X}+T_{2X}, \quad R_2 = T_{1Y}+T_{2Y}, \quad R_3 = T_{2Y}+T_{1Y} \tag{4.15}$$

where Q_1, Q_2 and Q_3 are projections of the principal vector and the principal moment of external forces divided by l, T_{iX} and T_{iY} ($i = 1, 2$) are the projections of the friction forces onto the coordinate axes (the abscissa axis passes through the point masses), w_1 and w_2 are projections of the accelerations of the points onto the ordinate axis, and w_3 is the projection of the accelerations onto the abscissa axis. We will divide the coordinate space $\Pi = \mathbb{R}^3$ into the following parts:

$$\Omega_1 = \{\pi|\pi_1^2+\pi_3^2<1, \pi_2>1\}, \quad \Omega_2 = \{\pi|\pi_1^2+\pi_3^2<1, \pi_2<-1\}$$
$$\Omega_3 = \{\pi|\pi_2^2+\pi_3^2<1, \pi_1>1\}, \quad \Omega_4 = \{\pi|\pi_2^2+\pi_3^2<1, \pi_1<-1\}$$
$$\Omega_5 = \left\{\pi\middle||\pi_1|<1, |\pi_2|<1, |\pi_3|<\sum_{i=1}^{2}\sqrt{1-\pi_i^2}\right\}$$
$$\Omega_6 = \{\pi|\pi_3>0\}\setminus\bigcup_{j=1}^{5}\Omega_j, \quad \Omega_7 = \{\pi|\pi_3<0\}\setminus\bigcup_{j=1}^{5}\Omega_j \tag{4.16}$$

The regions Ω_j ($j = 1, 2$) correspond to rotation of the rod about the point m_1, and here

$$T_{1X} = -f_1\pi_3, \quad T_{1Y} = -f_1\pi_1, \quad T_{2X} = 0, \quad T_{2Y} = -f_2\operatorname{sign}\pi_2$$
$$w_1 = w_3 = 0, \quad w_2 = (-1)^j(|\pi_2|-1), \quad f_i = \mu N_i, \quad i = 1, 2 \tag{4.17}$$

By analogy with formulae (4.17), coordinate mappings are plotted for regions Ω_j ($j = 3, 4$) corresponding to rotation about the point m_2. It is possible to check that, in each of these four regions, the mapping ψ is linear, and here inequalities (2.1) are satisfied.

In the region of stagnation Ω_5 we have

$$w = 0, \quad T_{iY} = -f_i\pi_i, \quad i = 1, 2$$

and here a unique definition of the components T_{iX} and T_{2X} is impossible in view of the static indeterminacy of the system. It can be stated that, in this region, $R = -Q$, and here the boundary of the image of the region of stagnation when mapping ψ is given by the equation

$$|Q_3| = \sqrt{f_1^2-Q_1^2}+\sqrt{f_2^2-Q_2^2}$$

Consequently, the problem is solved uniquely.

In region Ω_6 (sliding at both points) we assume that

$$w_3 = \pi_3-\chi(\pi_1)-\chi(\pi_2), \quad \chi(x) = \sqrt{1-\min\{x^2, 1\}}$$
$$w_i = \pi_i, \quad T_{iY} = -\frac{f_iw_i}{\sqrt{w_i^2+w_3^2}}, \quad T_{iX} = -\frac{f_iw_3}{\sqrt{w_i^2+w_3^2}}, \quad i = 1, 2 \tag{4.18}$$

Obviously, formulae (4.18) establish a one-to-one correspondence between w and π, the Jacobian of which is equal to unity. Therefore, when checking the conditions of the theorem, it is sufficient to examine the Jacob matrix $\|\partial Q/\partial w\|$. As follows from formulae (4.15) and (4.18)

$$Q_1 = \frac{\partial W}{\partial w_3}, \quad Q_2 = \frac{\partial W}{\partial w_2}+\frac{\partial W}{\partial w_1}, \quad Q_3 = \frac{\partial W}{\partial w_2}-\frac{\partial W}{\partial w_1}$$
$$W = \frac{1}{2}\sum_{i=1}^{2}(m_i(w_i^2+w_3^2)+2f_i\sqrt{w_i^2+w_3^2})$$

and here the function W is strictly convex. It follows from this that inequalities (2.1) are satisfied and the solution of system (4.15) is unique. Note that the conclusion given earlier was obtained[14] from the property of uniqueness of the extremum of the function $W - Qw$.

Example 2. A generalization of the previous example is the problem of the equilibrium of a rigid body resting in two small areas on a rough plane, the so-called "bench".[8] Besides gravity, a certain force is acting on the body, the line of action of which lies in the support plane. We will consider the problem in a restricted formulation, assuming that the height of the centre of mass of the bench, G, above the support remains unchanged and equal to h. Furthermore, the bench possesses two vertical planes of symmetry passing through the point G, one of which also contains the contact points. We will retain the former notation for the variables, and then the theorems of the momentum and moments (about the point G' – the projection of the centre of mass onto the support) are expressed by the relations

$$mw_3 = Q_1 + R_1, \quad \frac{1}{2}m(w_1 + w_2) = Q_2 + R_2, \quad \frac{1}{2}\alpha(w_2 - w_1) = Q_3 + R_3$$
$$R_1 = T_{1X} + T_{2X}, \quad R_2 = T_{1Y} + T_{2Y}, \quad R_3 = T_{2Y} - T_{1Y}, \quad \alpha = \frac{J}{l^2} \tag{4.19}$$

where m is the mass of the body and J is its central moment of inertia about the vertical.

In spite of the similarity of systems (4.15) and (4.19), they differ very considerably: whereas in the previous example the normal components of the reactions N_1 and N_2 were assumed to be specified, in the present case they depend on the generalized accelerations and are to be calculated. We will use the theorem of moments for the point G' projected onto the axis $G'Y'$, and we will also take into account the nature of the external forces:

$$mhw_3 = l(N_1 - N_2), \quad N_1 + N_2 = P \tag{4.20}$$

where P is the weight of the body. Solving system (4.20), we obtain

$$N_{1,2} = \frac{1}{2}(P \pm \beta w_3), \quad \beta = \frac{mh}{l} \tag{4.21}$$

Note that a difference between this and the previous examples appears only when $w_3 \neq 0$, which corresponds to sliding at both supports, and therefore, to check the correctness of the problem, it is sufficient to calculate the matrix $||\partial Q/\partial w||$ in region Ω_6 taking into account relations (4.21). The determinant of this matrix is a linear function of the parameter β:

$$\det\|\partial Q/\partial w\| = \Delta_0 + \Delta_1\beta$$

As shown by calculations, $\Delta_0 > 0$, and the quantity Δ_1 can be both positive and negative. When $\Delta_1 < 0$, for fairly large values of β system (4.19) may have a non-unique solution.

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